

Time Delay and Short-Range Scattering in Quantum Waveguides

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Abstract. Although many physical arguments account for using a modified definition of time delay in multichannel-type scattering processes, one can hardly find rigorous results on that issue in the literature. We try to fill in this gap by showing, both in an abstract setting and in a short-range case, the identity of the modified time delay and the Eisenbud-Wigner time delay in waveguides. In the short-range case we also obtain limiting absorption principles, state spectral properties of the total Hamiltonian, prove the existence of the wave operators and show an explicit formula for the S -matrix. The proofs rely on stationary and commutator methods.

1 Introduction and main results

This paper is concerned with time delay (defined in terms of sojourn times) in scattering theory for waveguides. Our main aim is to show that, as in N -body scattering and scattering by step potentials, one has to use a modified definition of time delay in order to prove its existence and its identity with the Eisenbud-Wigner time delay. We refer to [Mar75] for the treatment of this issue in the case of scattering with dissipative interactions.

Let us first recall the standard definition [JSM72] of time delay for an elastic two-body scattering process. Given a free Hamiltonian H_0 and a total Hamiltonian H such that the wave operators W^\pm exist and are complete, one defines for certain states φ and $r > 0$ two sojourn times, namely:

$$T_r^0(\varphi) := \int_{-\infty}^{\infty} dt \int_{|x| \leq r} d^3x |(e^{-itH_0}\varphi)(x)|^2 \quad (1.1)$$

and

$$T_r(\varphi) := \int_{-\infty}^{\infty} dt \int_{|x| \leq r} d^3x |(e^{-itH}W^-\varphi)(x)|^2. \quad (1.2)$$

The first number is interpreted as the time spent by the freely evolving state $e^{-itH_0}\varphi$ inside the ball $\mathcal{B}_r := \{x \in \mathbb{R}^3 : |x| \leq r\}$, whereas the second one is interpreted as the time spent by the associated scattering state $e^{-itH}W^-\varphi$ within the same region. Since $e^{-itH}W^-\varphi$ is asymptotically equal to $e^{-itH_0}\varphi$ as $t \rightarrow -\infty$, the difference

$$\tau_r^{\text{in}}(\varphi) := T_r(\varphi) - T_r^0(\varphi)$$

corresponds to the time delay of the scattering process with incoming state φ for the ball \mathcal{B}_r . The (global) time delay of the scattering process with incoming state φ

is, if it exists, the limit of $\tau_r^{\text{in}}(\varphi)$ as $r \rightarrow \infty$. For a suitable initial state φ and a sufficiently short-ranged interaction, it is known [AC87, ACS87] that this limit exists and is equal to the expectation value in the state φ of the Eisenbud-Wigner time delay operator.

If the scattering process associated to the pair $\{H_0, H\}$ is inelastic (typically of a N -body nature), then one has to modify the definition of time delay. The heuristic argument goes as follows. Due to the inelastic nature of the interaction, the expectation values of the momentum operator in the state $e^{-itH}W^-\varphi$ and in the state $e^{-itH_0}\varphi$ may converge to different constants as $t \rightarrow +\infty$. This would result in the divergence of the retardation (or advance) of the state $e^{-itH}W^-\varphi$ with respect to the state $e^{-itH_0}\varphi$. Similarly, if the incoming state φ is replaced by the outgoing state $S\varphi$, where S is the scattering operator, then the same divergence, but with an opposite sign, would occur as $t \rightarrow -\infty$. Therefore, in order to cancel both divergences out, $T_r(\varphi)$ should not be compared with the free sojourn time $T_r^0(\varphi)$, but with an effective free sojourn time involving both $T_r^0(\varphi)$ and $T_r^0(S\varphi)$. A symmetry argument [Mar81, Sec. V.(a)] leads naturally to the mean value $\frac{1}{2}[T_r^0(\varphi) + T_r^0(S\varphi)]$ for this effective time. Thus one ends up with the expression

$$\tau_r(\varphi) := T_r(\varphi) - \frac{1}{2} [T_r^0(\varphi) + T_r^0(S\varphi)] \quad (1.3)$$

for the time delay of the inelastic scattering process with incoming state φ for the ball \mathcal{B}_r . In the case of N -body scattering and step potential scattering, one can easily generalize the definition (1.3) to its multichannel counterpart [Smi60, BO79, Mar81].

Now consider a waveguide $\Omega := \Sigma \times \mathbb{R}$ with coordinates (x', x) , where Σ is a bounded open connected set in \mathbb{R}^{d-1} , $d \geq 2$. Let $H_0 := -\Delta_D^\Omega$ be the Dirichlet Laplacian in $L^2(\Omega)$ (equipped with the norm $\|\cdot\|$). Let H be a selfadjoint perturbation of H_0 such that the wave operators $W^\pm := \text{s-lim}_{t \rightarrow \pm\infty} e^{itH}e^{-itH_0}$ exist and are complete (so that the scattering operator $S := (W^+)^*W^-$ is unitary). Then the associated scattering process is globally elastic, but the kinetic energy along the x -axis is not conserved if the interaction is general enough. On the other hand, the waveguide counterparts of the sojourn times (1.1) and (1.2) must be

$$T_r^0(\varphi) := \int_{-\infty}^{\infty} dt \|F_r e^{-itH_0}\varphi\|^2 \quad (1.4)$$

and

$$T_r(\varphi) := \int_{-\infty}^{\infty} dt \|F_r e^{-itH}W^-\varphi\|^2, \quad (1.5)$$

where F_r denotes the projection onto the set of the states localized in the cylinder $\Omega_r := \Sigma \times [-r, r]$. Thus the sojourn times involve regions expanding in the x -direction, the axis along which the scattering process is inelastic. This explains why we have to use the formula (1.3) when defining time delay in waveguides. As in the N -body case, one can also write the time delay given by (1.3)–(1.5) in a multichannel way (see Remark 2.8).

Let us fix the notation and recall some properties of H_0 before giving a description of our results. \otimes (resp. \odot) stands for the closed (resp. algebraic) tensor product of Hilbert spaces or of operators. Given two Hilbert spaces \mathcal{H}_1 and \mathcal{H}_2 , we write $\mathcal{H}_1 \subset \mathcal{H}_2$ if \mathcal{H}_1 is continuously embedded in \mathcal{H}_2 and $\mathcal{H}_1 \simeq \mathcal{H}_2$ if \mathcal{H}_1 and \mathcal{H}_2 are isometric. $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ stands for the set of bounded operators from \mathcal{H}_1 to \mathcal{H}_2 with norm $\|\cdot\|_{\mathcal{H}_1 \rightarrow \mathcal{H}_2}$, and $\mathcal{B}(\mathcal{H}_1) := \mathcal{B}(\mathcal{H}_1, \mathcal{H}_1)$. $\|\cdot\|$ (resp. $\langle \cdot, \cdot \rangle$) denotes the norm (resp. scalar product) of the Hilbert space $\mathcal{H} := \mathbf{L}^2(\Omega) \simeq \mathbf{L}^2(\Sigma) \otimes \mathbf{L}^2(\mathbb{R})$. If there is no risk of confusion, the notations $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$ are also used for other spaces. Q (resp. P) stands for the position (resp. momentum) operator in $\mathbf{L}^2(\mathbb{R})$. $\mathbb{N} := \{0, 1, 2, \dots\}$ is the set of natural numbers. $\mathcal{H}^k(\Sigma)$, $k \in \mathbb{N}$, are the usual Sobolev spaces over Σ , and $\mathcal{H}_t^s(\mathbb{R}^n)$, $s, t \in \mathbb{R}$, $n \in \mathbb{N} \setminus \{0\}$, are the weighted Sobolev spaces over \mathbb{R}^n [ABG96, Sec. 4.1] (with the convention that $\mathcal{H}^s(\mathbb{R}^n) := \mathcal{H}_0^s(\mathbb{R}^n)$ and $\mathcal{H}_t(\mathbb{R}^n) := \mathcal{H}_t^0(\mathbb{R}^n)$). Given a selfadjoint operator A in a Hilbert space \mathcal{H} , we write $E^A(\cdot)$ for the spectral measure of A and $\mathcal{D}(A)$ for the domain of A endowed with its natural graph topology. $\chi_{[-r, r]}$ is the characteristic function for the interval $[-r, r]$ and $\langle \cdot \rangle := \sqrt{1 + |\cdot|^2}$.

The Dirichlet Laplacian $-\Delta_D^\Sigma$ in $\mathbf{L}^2(\Sigma)$ has a purely discrete spectrum $\mathcal{T} := \{\nu_\alpha\}_{\alpha \geq 1}$ consisting of eigenvalues $0 < \nu_1 < \nu_2 \leq \nu_3 \leq \dots$ repeated according to multiplicity. In particular $-\Delta_D^\Sigma$ admits the spectral decomposition $-\Delta_D^\Sigma = \sum_{\alpha \geq 1} \nu_\alpha \mathcal{P}_\alpha$, where \mathcal{P}_α is the one-dimensional orthogonal projection associated to ν_α . The Dirichlet Laplacian $-\Delta_D^\Omega$ can be written as $-\Delta_D^\Omega = -\Delta_D^\Sigma \otimes 1 + 1 \otimes P^2$, so that H_0 has a purely absolutely continuous spectrum coinciding with the interval $[\nu_1, \infty)$. Since S commutes with H_0 , S can be expressed as a direct integral of unitary operators $S(\lambda)$, $\lambda \geq \nu_1$, where $S(\lambda)$ acts in the fiber at energy λ in the spectral representation of H_0 (see Section 2.2). $S(\lambda)$ is called the S -matrix at energy λ .

Definition 1.1 *Let $\sigma_p(H)$ be the set of eigenvalues of H and $t \geq 0$, then*

$$\begin{aligned} \mathcal{D}_t^\mathbb{R} &:= \{\varphi \in \mathcal{H}_t(\mathbb{R}) : E^{P^2}(J)\varphi = \varphi \text{ for some compact set } J \text{ in } \mathbb{R} \setminus \{0\}\}, \\ \mathcal{D}_t^\Omega &:= \{\varphi \in \mathbf{L}^2(\Sigma) \otimes \mathcal{H}_t(\mathbb{R}) : E^{H_0}(J)\varphi = \varphi \text{ for some compact set } J \\ &\quad \text{in } (\nu_1, \infty) \setminus (\sigma_p(H) \cup \mathcal{T})\}. \end{aligned}$$

It is clear that $\mathcal{D}_t^\mathbb{R}$ is dense in $\mathbf{L}^2(\mathbb{R})$ and that $\mathcal{D}_{t_1}^\mathbb{R} \subset \mathcal{D}_{t_2}^\mathbb{R}$ if $t_1 \geq t_2$. The spaces \mathcal{D}_t^Ω also satisfy $\mathcal{D}_{t_1}^\Omega \subset \mathcal{D}_{t_2}^\Omega$ if $t_1 \geq t_2$, and \mathcal{D}_t^Ω is dense in \mathcal{H} .

We are in a position to state our results. In Section 2.3, we prove the following general existence criterion. It involves the Eisenbud-Wigner time delay operator $\tau_{\text{E-W}}$, which is the decomposable operator in the spectral representation of H_0 formally defined by the family

$$\tau_{\text{E-W}}(\lambda) := -iS(\lambda)^* \frac{dS(\lambda)}{d\lambda}, \quad \lambda \geq \nu_1.$$

Theorem 1.2 *Let $\Omega := \Sigma \times \mathbb{R}$, where Σ is a bounded open connected set in \mathbb{R}^{d-1} , $d \geq 2$. Consider a (two-body) scattering system in the Hilbert space $\mathcal{H} := \mathbf{L}^2(\Omega)$ with free Hamiltonian $H_0 := -\Delta_D^\Omega$ and total Hamiltonian H . Suppose that*

1. *For each $r > 0$ the projection F_r is locally H -smooth on $(\nu_1, \infty) \setminus (\sigma_p(H) \cup \mathcal{T})$.*
2. *The wave operators W^\pm exist and are complete.*

Let $\varphi \in \mathcal{D}_2^\Omega$ be such that $S\varphi \in \mathcal{D}_2^\Omega$ and

$$\|(W^- - 1)e^{-itH_0}\varphi\| \in \mathbf{L}^1((-\infty, 0), dt)$$

and

$$\|(W^+ - 1)e^{-itH_0}S\varphi\| \in \mathbf{L}^1((0, \infty), dt).$$

Then $\tau_r(\varphi)$ exists for each $r > 0$ and $\tau_r(\varphi)$ converges as $r \rightarrow \infty$ to a finite limit. If in addition the function $\lambda \mapsto S(\lambda)$ is strongly continuously differentiable on an open set $J \subset (\nu_1, \infty)$ such that $E^{H_0}(J)\varphi = \varphi$, then $\lim_{r \rightarrow \infty} \tau_r(\varphi) = \langle \varphi, \tau_{E-W}\varphi \rangle$.

Using the stationary formalism of [Kur73] and the commutator methods of [ABG96], we prove in Section 3.1 some results concerning short-range scattering theory in waveguides. In Theorem 3.4 we obtain limiting absorption principles and state spectral properties of the total Hamiltonian. In Corollary 3.5 we show the existence and the completeness of the wave operators (see also [DEŠ95] for a two-dimensional situation). In addition we prove a result on the norm differentiability of the S -matrix (Proposition 3.8) which relies on an explicit formula for the S -matrix (Lemma 3.7). In Section 3.2, we use the results of Section 3.1 to find sufficient conditions under which the hypotheses of Theorem 1.2 are satisfied (see Theorem 3.11 for the precise statement):

Theorem 1.3 *Let $H := H_0 + V$, where V decays as $|x|^{-\kappa}$, $\kappa > 4$, at infinity. Then there exists a dense set \mathcal{E} such that, for each $\varphi \in \mathcal{E}$, $\tau_r(\varphi)$ exists for all $r > 0$ and $\tau_r(\varphi)$ converges as $r \rightarrow \infty$ to a finite limit equal to $\langle \varphi, \tau_{E-W}\varphi \rangle$.*

Remark 1.4 *A comparison with the corresponding theorem [ACS87, Prop. 4] for scattering in \mathbb{R}^d , shows us that potentials decaying as $|x|^{-\kappa}$, $\kappa > 2$, at infinity may also be treated. This could certainly be done by adapting results on the mapping properties of the scattering operator (e.g., [ACS87, JN92]) to the waveguide case. However, since these properties deserve a study on their own, we prefer not to use them in the present paper.*

We finally mention Lemma 2.4 which establishes some regularity properties of the trace-type operator associated to the spectral transformation for H_0 .

2 General existence of time delay in waveguides

2.1 Preliminaries

In the sequel we give sufficient conditions for the existence of the time delay in Ω_r . Then we show that the (global) time delay, if it exists, is expressed in terms

of the limit of an auxiliary time. We start by recalling some facts which will be freely used throughout the paper.

The one-dimensional Fourier transform \mathcal{F} is a topological isomorphism of $\mathcal{H}_t^s(\mathbb{R})$ onto $\mathcal{H}_s^t(\mathbb{R})$ for any $s, t \in \mathbb{R}$. Given two separable Hilbert spaces \mathcal{H}_1 and \mathcal{H}_2 one has the relation $(\mathcal{H}_1 \otimes \mathcal{H}_2)^* \simeq \mathcal{H}_1^* \otimes \mathcal{H}_2^*$ for their adjoint spaces. Furthermore, if 1 is the identity operator in \mathcal{H}_1 and A a selfadjoint operator in \mathcal{H}_2 , then one has the identity $\mathcal{D}(1 \otimes A) \simeq \mathcal{H}_1 \otimes \mathcal{D}(A)$. If $\mathcal{H}_1, \mathcal{H}_2, \mathcal{K}_1, \mathcal{K}_2$ are Hilbert spaces and $A_i \in \mathcal{B}(\mathcal{H}_i, \mathcal{K}_i)$ ($i = 1, 2$), then $A_1 \otimes A_2 \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2, \mathcal{K}_1 \otimes \mathcal{K}_2)$.

Remark 2.1 Since $H_0 = -\Delta_D^\Sigma \otimes 1 + 1 \otimes P^2$, the domain of H_0 has the following form [BG92, Sec. 3]:

$$\mathcal{D}(H_0) = [\mathcal{D}(-\Delta_D^\Sigma) \otimes \mathbf{L}^2(\mathbb{R})] \cap [\mathbf{L}^2(\Sigma) \otimes \mathcal{H}^2(\mathbb{R})].$$

The set $\mathcal{D}(H_0)$ is endowed with the intersection topology, so that it is a Hilbert space. The spectral measure of H_0 admits the tensorial decomposition [Wei80, Ex. 8.21]:

$$E^{H_0}(\cdot) = \sum_{\alpha \geq 1} \mathcal{P}_\alpha \otimes E^{P^2 + \nu_\alpha}(\cdot).$$

Hence the equality

$$e^{itH_0} = \sum_{\alpha \geq 1} \mathcal{P}_\alpha \otimes e^{it(P^2 + \nu_\alpha)} \quad (2.1)$$

holds in the sense of the strong convergence. Furthermore each $\varphi \in \mathcal{D}_t^\Omega$ is a finite sum of vectors $\varphi_\alpha^\Sigma \otimes \varphi_\alpha^\mathbb{R}$, where $\varphi_\alpha^\Sigma \in \mathcal{P}_\alpha \mathbf{L}^2(\Sigma)$ and $\varphi_\alpha^\mathbb{R} \in \mathcal{D}_t^\mathbb{R}$.

For each $r > 0$, we define the auxiliary time $\tau_r^{\text{free}}(\varphi)$ by

$$\begin{aligned} \tau_r^{\text{free}}(\varphi) := \frac{1}{2} \Big\{ & \int_{-\infty}^0 dt \left[\|F_r e^{-itH_0} \varphi\|^2 - \|F_r e^{-itH_0} S\varphi\|^2 \right] \\ & + \int_0^\infty dt \left[\|F_r e^{-itH_0} S\varphi\|^2 - \|F_r e^{-itH_0} \varphi\|^2 \right] \Big\}. \end{aligned}$$

The superscript “free” makes reference to the fact that the formula for $\tau_r^{\text{free}}(\varphi)$ involves only the free evolution of the vectors φ and $S\varphi$.

Lemma 2.2 Suppose that the hypotheses 1 and 2 of Theorem 1.2 hold and let $r > 0$, $\varphi \in \mathcal{D}_0^\Omega$. Then

- (a) $\|F_r e^{-itH_0} \varphi\|$ belongs to $\mathbf{L}^2(\mathbb{R}, dt)$,
- (b) $\|F_r e^{-itH_0} S\varphi\|$ belongs to $\mathbf{L}^2(\mathbb{R}, dt)$,
- (c) $\|F_r e^{-itH} W^- \varphi\|$ belongs to $\mathbf{L}^2(\mathbb{R}, dt)$,
- (d) $\tau_r(\varphi)$ and $\tau_r^{\text{free}}(\varphi)$ exist.

Proof. Since $F_r = 1 \otimes \chi_{[-r,r]}(Q)$, the point (a) follows from Remark 2.1 and the local smoothness [Lav73, Thm. 1] of $\chi_{[-r,r]}(Q)$ with respect to P^2 . Since S and $E^{H_0}(\cdot)$ commute, the statement (b) can be shown as (a). The point (c) follows from the intertwining relation $E^H(\cdot)W^\pm = W^\pm E^{H_0}(\cdot)$ and the fact that F_r is locally H -smooth on $(\nu_1, \infty) \setminus (\sigma_p(H) \cup \mathcal{T})$. The last statement is a consequence of points (a), (b) and (c). \square

The following result can be easily deduced from the proof of [AC87, Prop. 2].

Lemma 2.3 *Suppose that the hypotheses 1 and 2 of Theorem 1.2 hold and let $\varphi \in \mathcal{D}_0^\Omega$ be such that*

$$\|(W^- - 1)e^{-itH_0}\varphi\| \in L^1((-\infty, 0), dt)$$

and

$$\|(W^+ - 1)e^{-itH_0}S\varphi\| \in L^1((0, \infty), dt).$$

Then one has the equality

$$\lim_{r \rightarrow \infty} \tau_r(\varphi) = \lim_{r \rightarrow \infty} \tau_r^{\text{free}}(\varphi). \quad (2.2)$$

We emphasize that the equation (2.2) should be interpreted as follows: if one of the two limits exists, then so does the other one, and the two limits are equal.

2.2 Spectral decomposition and trace-type operator

We now gather some results on the spectral transformation for H_0 and on the associated trace-type operator. We begin with the definition of the trace-type operator. $\mathcal{H}(\lambda)$ denotes the fibre at energy $\lambda \geq \nu_1$ in the spectral representation of H_0 :

$$\mathcal{H}(\lambda) := \bigoplus_{\alpha \in \mathbb{N}(\lambda)} \{\mathcal{P}_\alpha L^2(\Sigma) \oplus \mathcal{P}_\alpha L^2(\Sigma)\},$$

where $\mathbb{N}(\lambda) := \{\alpha \in \mathbb{N} \setminus \{0\} : \nu_\alpha \leq \lambda\}$. Since $\mathcal{H}(\lambda)$ is naturally embedded in

$$\mathcal{H}(\infty) := \bigoplus_{\alpha \geq 1} \{\mathcal{P}_\alpha L^2(\Sigma) \oplus \mathcal{P}_\alpha L^2(\Sigma)\},$$

we shall sometimes write $\mathcal{H}(\infty)$ instead of $\mathcal{H}(\lambda)$. For $\xi \in \mathbb{R}$, let $\gamma(\xi): \mathcal{S}(\mathbb{R}) \rightarrow \mathbb{C}$ be the trace operator given by $\gamma(\xi)\varphi := \varphi(\xi)$. Then, for $\lambda \in (\nu_1, \infty) \setminus \mathcal{T}$, we define the trace-type operator $T(\lambda): L^2(\Sigma) \odot \mathcal{S}(\mathbb{R}) \rightarrow \mathcal{H}(\lambda)$ by

$$[T(\lambda)\varphi]_\alpha := (\lambda - \nu_\alpha)^{-1/4} \left\{ [\mathcal{P}_\alpha \otimes \gamma(-\sqrt{\lambda - \nu_\alpha})]\varphi, [\mathcal{P}_\alpha \otimes \gamma(\sqrt{\lambda - \nu_\alpha})]\varphi \right\}. \quad (2.3)$$

In the next lemma we show some regularity properties of the operator $T(\lambda)$. The proof can be found in the appendix.

Lemma 2.4 *Let $t \in \mathbb{R}$. Then*

- (a) *For any $\lambda \in (\nu_1, \infty) \setminus \mathcal{T}$ and $s > 1/2$, the operator $T(\lambda)$ extends to an element of $\mathcal{B}(\mathbf{L}^2(\Sigma) \otimes \mathcal{H}_t^s(\mathbb{R}), \mathcal{H}(\infty))$.*
- (b) *For any $s > 1/2$, the function $T: (\nu_1, \infty) \setminus \mathcal{T} \rightarrow \mathcal{B}(\mathbf{L}^2(\Sigma) \otimes \mathcal{H}_t^s(\mathbb{R}), \mathcal{H}(\infty))$ is locally Hölder continuous.*
- (c) *For any $s > n + 1/2$, $n \in \mathbb{N}$, the function $\lambda \mapsto T(\lambda)$ is n times continuously differentiable as a map from $(\nu_1, \infty) \setminus \mathcal{T}$ to $\mathcal{B}(\mathbf{L}^2(\Sigma) \otimes \mathcal{H}_t^s(\mathbb{R}), \mathcal{H}(\infty))$.*

We give now the spectral transformation for H_0 in terms of the operators $T(\lambda)$.

Proposition 2.5 *The mapping $\mathcal{U}: \mathcal{H} \rightarrow \int_{[\nu_1, \infty)}^{\oplus} d\lambda \mathcal{H}(\lambda)$, defined by*

$$(\mathcal{U}\varphi)(\lambda) := 2^{-1/2} T(\lambda)(1 \otimes \mathcal{F})\varphi \quad (2.4)$$

for all $\varphi \in \mathbf{L}^2(\Sigma) \odot \mathcal{S}(\mathbb{R})$, $\lambda \in (\nu_1, \infty) \setminus \mathcal{T}$, is unitary and

$$\mathcal{U} H_0 \mathcal{U}^* = \int_{[\nu_1, \infty)}^{\oplus} d\lambda \lambda.$$

Proof. A direct calculation shows that $\|\mathcal{U}\varphi\| = \|\varphi\|$ for all $\varphi \in \mathbf{L}^2(\Sigma) \odot \mathcal{S}(\mathbb{R})$. Since $\mathbf{L}^2(\Sigma) \odot \mathcal{S}(\mathbb{R})$ is dense in \mathcal{H} , this implies that \mathcal{U} is an isometry. Furthermore, for any $\psi \equiv \{\psi_\alpha^-(\lambda), \psi_\alpha^+(\lambda)\} \in \int_{[\nu_1, \infty)}^{\oplus} d\lambda \mathcal{H}(\lambda)$, one can check that

$$\mathcal{U}^* \psi = (1 \otimes \mathcal{F}^*) \tilde{\psi} \quad \text{where} \quad \tilde{\psi}(\cdot, \xi) := \begin{cases} \sqrt{2|\xi|} \sum_{\alpha \geq 1} \psi_\alpha^-(\xi^2 + \nu_\alpha) & \text{if } \xi < 0 \\ \sqrt{2|\xi|} \sum_{\alpha \geq 1} \psi_\alpha^+(\xi^2 + \nu_\alpha) & \text{if } \xi \geq 0, \end{cases} \quad (2.5)$$

so that $\|\mathcal{U}^* \psi\| = \|\psi\|$. Hence \mathcal{U} is unitary. The second statement follows by using (2.3) and (2.4). \square

Since the scattering operator S commutes with H_0 , it follows by Proposition 2.5 that S admits the direct integral decomposition

$$\mathcal{U} S \mathcal{U}^* = \int_{[\nu_1, \infty)}^{\oplus} d\lambda S(\lambda),$$

where $S(\lambda)$ (the S -matrix at energy λ) is an operator acting unitarily in $\mathcal{H}(\lambda)$.

2.3 Existence theorem

In the present section we shall give the proof of Theorem 1.2. We first prove an asymptotic formula involving

$$D_0 := \frac{1}{2} (P^{-1}Q + QP^{-1}),$$

which is a well-defined symmetric operator on $\mathcal{D}_1^{\mathbb{R}}$.

Proposition 2.6

(a) Suppose that the hypothesis 2 of Theorem 1.2 holds and let $\varphi \in \mathcal{D}_0^\Omega$. Then

$$\tau_r^{\text{free}}(\varphi) = \frac{1}{2} \int_0^\infty dt \left\langle S\varphi, \left[1 \otimes \left(e^{itP^2} \chi_{[-r,r]}(Q) e^{-itP^2} - e^{-itP^2} \chi_{[-r,r]}(Q) e^{itP^2} \right), S \right] \varphi \right\rangle.$$

(b) For all $\varphi, \psi \in \mathcal{D}_2^\mathbb{R}$

$$\begin{aligned} & \lim_{r \rightarrow \infty} \int_0^\infty dt \left\langle \varphi, \left[e^{itP^2} \chi_{[-r,r]}(Q) e^{-itP^2} - e^{-itP^2} \chi_{[-r,r]}(Q) e^{itP^2} \right] \psi \right\rangle \\ &= - \langle \varphi, D_0 \psi \rangle. \end{aligned} \quad (2.6)$$

(c) Suppose that the hypothesis 2 of Theorem 1.2 holds and let $\varphi \in \mathcal{D}_2^\Omega$ be such that $S\varphi \in \mathcal{D}_2^\Omega$. Then

$$\lim_{r \rightarrow \infty} \tau_r^{\text{free}}(\varphi) = -\frac{1}{2} \langle \varphi, S^*[1 \otimes D_0, S]\varphi \rangle. \quad (2.7)$$

Proof. (a) Due to (2.1), one has the equality

$$e^{itH_0} F_r e^{-itH_0} = 1 \otimes e^{itP^2} \chi_{[-r,r]}(Q) e^{-itP^2}.$$

This together with the unitarity of the scattering operator implies the claim.

(b) (i) It is sufficient to prove (2.6) for $\varphi = \psi$, the case $\varphi \neq \psi$ being obtained by means of the polarization identity.

For any $f \in L^\infty(\mathbb{R})$ and $t > 0$ one has [AJS77, Eq. (13.4)]

$$e^{itP^2} f(Q) e^{-itP^2} = Z_{1/4t}^* f(2tP) Z_{1/4t},$$

where $Z_\tau := e^{i\tau Q^2}$. This together with the change of variables $\mu := r(2t)^{-1}$ and $\nu := (2r)^{-1}$ leads to the equality

$$\begin{aligned} & \int_0^\infty dt \left\langle \varphi, \left[e^{itP^2} \chi_{[-r,r]}(Q) e^{-itP^2} - e^{-itP^2} \chi_{[-r,r]}(Q) e^{itP^2} \right] \varphi \right\rangle \\ &= \frac{1}{4} \int_0^\infty \frac{d\mu}{\nu\mu^2} \left\langle \varphi, \left[Z_{\nu\mu}^* \chi_{[-\mu,\mu]}(P) Z_{\nu\mu} - Z_{\nu\mu} \chi_{[-\mu,\mu]}(P) Z_{\nu\mu}^* \right] \varphi \right\rangle. \end{aligned} \quad (2.8)$$

Hence the l.h.s. of (2.6) (for $\varphi = \psi$) can be written as

$$\begin{aligned} K_\infty(\varphi) &:= \lim_{\nu \searrow 0} \frac{1}{4} \int_0^\infty \frac{d\mu}{\nu\mu^2} \left\langle \varphi, \left[Z_{\nu\mu}^* \chi_{[-\mu,\mu]}(P) Z_{\nu\mu} - \chi_{[-\mu,\mu]}(P) \right. \right. \\ &\quad \left. \left. + \chi_{[-\mu,\mu]}(P) - Z_{\nu\mu} \chi_{[-\mu,\mu]}(P) Z_{\nu\mu}^* \right] \varphi \right\rangle. \end{aligned} \quad (2.9)$$

(ii) To prove the statement, we shall show that one may interchange the limit and the integral in (2.9), by invoking the Lebesgue dominated convergence theorem. This will be done in (iii) below. If one assumes the result for the moment, then a direct calculation as in [AC87, Sec. 2] leads to the desired equality, that is

$$\begin{aligned} K_\infty(\varphi) &= \frac{1}{4} \int_0^\infty \frac{d\mu}{\mu^2} \frac{d}{d\nu} \left\langle \varphi, [Z_{\nu\mu}^* \chi_{[-\mu, \mu]}(P) Z_{\nu\mu} - Z_{\nu\mu} \chi_{[-\mu, \mu]}(P) Z_{\nu\mu}^*] \varphi \right\rangle \Big|_{\nu=0} \\ &= -\langle \varphi, D_0 \varphi \rangle \end{aligned}$$

if $\varphi \in \mathcal{D}_2^{\mathbb{R}}$.

(iii) It remains to prove the applicability of the Lebesgue dominated convergence theorem to (2.9). For this we rewrite (2.8) (which is equivalent to (2.9)) as

$$\begin{aligned} K_\infty(\varphi) &= \lim_{\nu \searrow 0} \frac{1}{4} \int_0^\infty \frac{d\mu}{\mu} \left[\left\langle \chi_{[-\mu, \mu]}(P) Z_{\nu\mu} \varphi, \frac{Z_{\nu\mu} - Z_{\nu\mu}^*}{\nu\mu} \varphi \right\rangle \right. \\ &\quad \left. + \left\langle \frac{Z_{\nu\mu} - Z_{\nu\mu}^*}{\nu\mu} \varphi, \chi_{[-\mu, \mu]}(P) Z_{\nu\mu}^* \varphi \right\rangle \right]. \end{aligned} \quad (2.10)$$

Since $\tau^{-1}(Z_\tau - Z_\tau^*)\varphi$ converges strongly to $2iQ^2\varphi$ as $\tau \rightarrow 0$, we may choose a number $\delta > 0$ such that $\|\tau^{-1}(Z_\tau - Z_\tau^*)\varphi\| \leq 3\|Q^2\varphi\|$ for all $\tau \in [-\delta, \delta]$. We then have

$$\left\| \frac{1}{\nu\mu} (Z_{\nu\mu} - Z_{\nu\mu}^*) \varphi \right\| \leq \begin{cases} 3\|Q^2\varphi\| & \text{if } \nu\mu \leq \delta \\ \frac{2}{\delta}\|\varphi\| & \text{if } \nu\mu \geq \delta. \end{cases} \quad (2.11)$$

Let $\ell \in (0, 1/2)$, then $|P|^{-\ell} \langle Q \rangle^{-2}$ belongs to $\mathcal{B}(\mathbf{L}^2(\mathbb{R}))$ (after exchanging the role of P and Q , this follows from the fact that $|Q|^{-\ell}$ is P^2 -bounded [Amr81, Prop. 2.28]), and

$$|\mu^{-1} \xi|^\ell \chi_{[-\mu, \mu]}(\xi) \leq \chi_{[-\mu, \mu]}(\xi) \leq 1$$

for all $\xi \in \mathbb{R}$. Thus one has the estimate

$$\begin{aligned} \mu^{-1} \|\chi_{[-\mu, \mu]}(P) Z_{\pm\nu\mu} \varphi\| &= \mu^{\ell-1} \|\mu^{-1} |P|^\ell \chi_{[-\mu, \mu]}(P) |P|^{-\ell} \langle Q \rangle^{-2} Z_{\pm\nu\mu} \langle Q \rangle^2 \varphi\| \\ &\leq \text{Const. } \mu^{\ell-1} \|\langle Q \rangle^2 \varphi\|. \end{aligned} \quad (2.12)$$

Hence (2.11) and (2.12) imply that the integrand in (2.10) is bounded by a function in $\mathbf{L}_{\text{loc}}^1((0, \infty), d\mu)$, which is sufficient for applying the Lebesgue dominated convergence theorem on any finite interval $[0, \mu_0]$.

Since the case $\mu \rightarrow \infty$ can be treated as in [AC87, Sec. 2], this concludes the proof of the statement.

(c) This is a consequence of Remark 2.1 and points (a) and (b). \square

Remark 2.7 We know from Section 2.2 that \mathcal{H} can be identified with the direct integral $\int_{[\nu_1, \infty)}^\oplus d\lambda \mathcal{H}(\lambda)$, where H_0 acts as the multiplication operator by λ . So one may write $\varphi(\lambda)$ for the component of $\varphi \in \mathcal{H}$ at energy λ and $\langle \cdot, \cdot \rangle_{\mathcal{H}(\lambda)}$ for the scalar

product in $\mathcal{H}(\lambda)$. A direct calculation using (2.3)–(2.5) shows that $1 \otimes D_0 = 2i \frac{d}{d\lambda}$ in the spectral representation of H_0 . On the other hand $\varphi \in \mathcal{D}(1 \otimes D_0^2)$ if $\varphi \in \mathcal{D}_2^\Omega$. Therefore if $\varphi \in \mathcal{D}_2^\Omega$, then the function $\lambda \mapsto \varphi(\lambda)$ is continuously differentiable on each interval $(\nu_\alpha, \nu_{\alpha+1})$. As a consequence, if $\varphi \in \mathcal{D}_2^\Omega$ is such that $S\varphi \in \mathcal{D}_2^\Omega$, and if the function $\lambda \mapsto S(\lambda)$ is strongly continuously differentiable on the support of $\varphi(\cdot)$, then one gets from (2.7) the equalities

$$\lim_{r \rightarrow \infty} \tau_r^{\text{free}}(\varphi) = -i \int_{\nu_1}^{\infty} d\lambda \left\langle \varphi(\lambda), S(\lambda)^* \left[\frac{dS(\lambda)}{d\lambda} \right] \varphi(\lambda) \right\rangle_{\mathcal{H}(\lambda)} \equiv \langle \varphi, \tau_{\text{E-W}} \varphi \rangle. \quad (2.13)$$

Provided that (2.2) holds, (2.13) expresses the identity of the (global) time delay and the Eisenbud-Wigner time delay in waveguides.

Theorem 1.2 is a direct consequence of Lemma 2.2, Lemma 2.3, Proposition 2.6 and Remark 2.7.

Remark 2.8 The S -matrix at energy λ can be written as the double sum

$$S(\lambda) = \sum_{\beta, \alpha \in \mathbb{N}(\lambda)} S_{\beta\alpha}(\lambda),$$

where $S_{\beta\alpha}(\lambda) := [\mathcal{W}(\mathcal{P}_\beta \otimes 1)S(\mathcal{P}_\alpha \otimes 1)\mathcal{W}^*](\lambda)$. Therefore if φ_α is a vector in $(\mathcal{P}_\alpha \otimes 1)\mathcal{H}$ satisfying the hypotheses of Theorem 1.2, then a simple calculation shows that (2.13) is equivalent to

$$\lim_{r \rightarrow \infty} \tau_r^{\text{free}}(\varphi_\alpha) = -i \int_{\nu_1}^{\infty} d\lambda \left\langle \varphi_\alpha(\lambda), \sum_{\beta \in \mathbb{N}(\lambda)} S_{\beta\alpha}(\lambda)^* \left[\frac{dS_{\beta\alpha}(\lambda)}{d\lambda} \right] \varphi_\alpha(\lambda) \right\rangle_{\mathcal{H}(\lambda)}. \quad (2.14)$$

This equation admits a natural interpretation: if each subspace $(\mathcal{P}_\alpha \otimes 1)\mathcal{H}$ is seen as a channel Hilbert space, then (2.14) can be considered as a multichannel formulation in waveguides of the identity of the (global) time delay and the Eisenbud-Wigner time delay for an incoming state in channel α .

3 Time delay in waveguides: the short-range case

3.1 Short-range scattering in waveguides

In this section we collect some results on the scattering theory for the pair $\{H_0, H\}$ in the case $H := H_0 + V$, where V is a short-range potential satisfying the following condition:

Assumption 3.1 V is a multiplication operator by a real-valued measurable function on Ω such that V defines a compact operator from $\mathcal{D}(H_0)$ to \mathcal{H} and a bounded operator from $\mathbb{L}^2(\Sigma) \otimes \mathcal{H}^2(\mathbb{R})$ to $\mathbb{L}^2(\Sigma) \otimes \mathcal{H}_\kappa(\mathbb{R})$ for some $\kappa > 1$.

By using duality, interpolation and the fact that V commutes with the operator $1 \otimes \langle Q \rangle^t$, $t \in \mathbb{R}$, one shows that V also defines a bounded operator from $L^2(\Sigma) \otimes \mathcal{H}_t^{2s}(\mathbb{R})$ to $L^2(\Sigma) \otimes \mathcal{H}_{t+\kappa}^{2(s-1)}(\mathbb{R})$ for any $s \in [0, 1]$, $t \in \mathbb{R}$.

If V satisfies Assumption 3.1, then the operator H is selfadjoint on $\mathcal{D}(H) = \mathcal{D}(H_0)$, $(H + i)^{-1} - (H_0 + i)^{-1}$ is compact and $\sigma_{\text{ess}}(H) = \sigma_{\text{ess}}(H_0) = [\nu_1, \infty)$. In order to get more informations on H , we shall apply the conjugate operator method. We refer to [ABG96] for the definitions of the regularity classes appearing in the sequel, and for more explanations on the conjugate operator method.

For $\varepsilon \in (0, 1)$, we choose a function $\vartheta \in C_0^\infty((\varepsilon, \infty))$ and define $F: \mathbb{R} \rightarrow \mathbb{R}$ by

$$F(x) := \begin{cases} \frac{1}{2x} \vartheta(x^2) & \text{if } x \in (-\infty, -\sqrt{\varepsilon}) \cup (\sqrt{\varepsilon}, \infty) \\ 0 & \text{otherwise.} \end{cases}$$

We first introduce the operator $A_{\parallel} := F(P)Q + \frac{i}{2}F'(P)$ acting on $\mathcal{S}(\mathbb{R})$. A_{\parallel} has the following properties [ABG96, Lemma 7.6.4]: A_{\parallel} is essentially selfadjoint, the group $\{e^{i\tau A_{\parallel}}\}_{\tau \in \mathbb{R}}$ leaves $\mathcal{D}(-\Delta^{\mathbb{R}}) = \mathcal{H}^2(\mathbb{R})$ invariant, $-\Delta^{\mathbb{R}}$ is of class $C^\infty(A_{\parallel})$ and A_{\parallel} is strictly conjugate to $-\Delta^{\mathbb{R}}$ on $(-\infty, 0) \cup I_{\vartheta}$, where $I_{\vartheta} := \{u \in (\varepsilon, \infty) : \vartheta(u) = 1\}$. Now let $A := 1 \otimes A_{\parallel}$. It turns out that H_0 has many regularity properties with respect to A , namely (see [BG92, Sec. 3]) $\{e^{i\tau A}\}_{\tau \in \mathbb{R}}$ is a C_0 -group in $\mathcal{D}(H_0)$, H_0 is of class $C^\infty(A)$ and A is strictly conjugate to H_0 on $(-\infty, \nu_1) \cup J_{\vartheta}$, where J_{ϑ} is a bounded open set in $(\nu_1, \infty) \setminus \mathcal{T}$ depending on I_{ϑ} . The exact nature of J_{ϑ} can be explicitly deduced from that of I_{ϑ} by using the formula [BG92, Eq. (3.8)], which relates the Mourre estimate for $-\Delta^{\mathbb{R}}$ to the Mourre estimate for H_0 . In our case it is enough to note that, given any compact set K in $\mathbb{R} \setminus \mathcal{T}$, there exist $\varepsilon \in (0, 1)$ and $\vartheta \in C_0^\infty((\varepsilon, \infty))$ such that K is contained in $(-\infty, \nu_1) \cup J_{\vartheta}$.

Now we prove that V also satisfies regularity conditions with respect to A . Given an operator B in \mathcal{H} and a Hilbert space $\mathcal{G} \subset \mathcal{H}$, we write $\mathcal{D}(B; \mathcal{G}) := \{\varphi \in \mathcal{D}(B) \cap \mathcal{G} : B\varphi \in \mathcal{G}\}$ for the domain of B in \mathcal{G} .

Lemma 3.2 *Let V satisfy Assumption 3.1. Then*

- (a) *V is of class $\mathcal{C}^{1,1}(A; \mathcal{D}(H_0), \mathcal{D}(H_0)^*)$.*
- (b) *The operators $[H_0, A]$ and $[H, A]$, which a priori only belong to $\mathcal{B}(\mathcal{D}(H_0), \mathcal{D}(H_0)^*)$, are such that $[H_0, A] \in \mathcal{B}(\mathcal{D}(H_0))$ and $[H, A] \in \mathcal{B}(\mathcal{D}(H_0), \mathcal{H})$.*

Proof. (a) We use the criterion [ABG96, Thm. 7.5.8] to prove the statement. The three conditions needed for that theorem are obtained in points (i), (ii) and (iii) below.

(i) Let $\Lambda := 1 \otimes \langle Q \rangle$. Since $\{e^{i\tau \langle Q \rangle}\}_{\tau \in \mathbb{R}}$ is a polynomially bounded C_0 -group in $\mathcal{H}^2(\mathbb{R})$ [ABG96, Sec. 7.6.3], a direct calculation using the tensorial decomposition of H_0 (see Remark 2.1) shows that $\{e^{i\tau \Lambda}\}_{\tau \in \mathbb{R}}$ is a polynomially bounded C_0 -group in $\mathcal{D}(H_0)$.

(ii) Since $\{e^{i\tau A}\}_{\tau \in \mathbb{R}}$ is a C_0 -group in $\mathcal{D}(H_0)$, there exists $r > 0$ such that $-ir$ belongs to the resolvent set of A (considered as an operator in $\mathcal{D}(H_0)$). In

particular, the operator $(A + ir)^{-1} = -i \int_0^\infty d\tau e^{-r\tau} e^{i\tau A}$ is a homeomorphism from $\mathcal{D}(H_0)$ onto $\mathcal{D}(A; \mathcal{D}(H_0))$ (both domains being endowed with their natural graph topology). Therefore any set \mathcal{E} of the form $(A + ir)^{-1} \mathcal{D}$, with \mathcal{D} dense in $\mathcal{D}(H_0)$, is dense in $\mathcal{D}(A; \mathcal{D}(H_0))$. Let us take $\mathcal{D} := \{\varphi_\alpha\} \odot \mathcal{S}(\mathbb{R})$, where $\{\varphi_\alpha\}$ is the set of eigenvectors of $-\Delta_{\mathbb{D}}^\Sigma$ (since $H_0 \upharpoonright \mathcal{D}$ is essentially selfadjoint, \mathcal{D} is dense in $\mathcal{D}(H_0)$). A vector ψ in \mathcal{E} is of the form $\psi = -i \sum_{\alpha \leq \text{Const.}} \varphi_\alpha \otimes \int_0^\infty d\tau e^{-r\tau} e^{i\tau A_{\text{II}}} \eta_\alpha$, where $(\varphi_\alpha, \eta_\alpha) \in \{\varphi_\alpha\} \times \mathcal{S}(\mathbb{R})$ and the integral converges in $\mathcal{H}^2(\mathbb{R})$. Since $\langle Q \rangle^{-2} \in \mathcal{B}(\mathcal{L}^2(\mathbb{R}))$ and $A_{\text{II}} \eta_\alpha \in \mathcal{S}(\mathbb{R})$, the vector

$$\tilde{\psi} := -i \sum_{\alpha \leq \text{Const.}} \varphi_\alpha \otimes \int_0^\infty d\tau e^{-r\tau} \langle Q \rangle^{-2} e^{i\tau A_{\text{II}}} A_{\text{II}} \eta_\alpha$$

belongs to \mathcal{H} . Furthermore $\tilde{\psi} = \Lambda^{-2} A \psi$ and $\Lambda^{-2} A \psi \in \mathcal{D}(H_0)$. Since $e^{i\tau A_{\text{II}}} \eta_\alpha \in \mathcal{S}(\mathbb{R})$ [ABG96, Prop. 4.2.4], one can use commutator expansions to get the equality

$$\| \langle Q \rangle^{-2} e^{i\tau A_{\text{II}}} A_{\text{II}} \eta_\alpha - S_1 \langle Q \rangle^{-1} e^{i\tau A_{\text{II}}} \eta_\alpha \|_{\mathcal{H}^2(\mathbb{R})} = 0$$

for some operator $S_1 \in \mathcal{B}(\mathcal{H}^2(\mathbb{R}))$. This implies that

$$\| \Lambda^{-2} A \psi - (1 \otimes S_1) \Lambda^{-1} \psi \|_{\mathcal{D}(H_0)} = 0 \quad (3.1)$$

for $\psi \in \mathcal{E}$. Since the operators $1 \otimes S_1$ and Λ^{-1} belong to $\mathcal{B}(\mathcal{D}(H_0))$ and \mathcal{E} is dense in $\mathcal{D}(A; \mathcal{D}(H_0))$, (3.1) even holds for $\psi \in \mathcal{D}(A; \mathcal{D}(H_0))$. Thus, for each $\psi \in \mathcal{D}(A^2; \mathcal{D}(H_0))$, one gets

$$\| \Lambda^{-2} A^2 \psi - (1 \otimes S_1) \Lambda^{-1} A \psi \|_{\mathcal{D}(H_0)} = \| (\Lambda^{-2} A) A \psi - (1 \otimes S_1) \Lambda^{-1} A \psi \|_{\mathcal{D}(H_0)} = 0.$$

Using an argument similar to the one leading to (3.1), one shows that

$$\| \Lambda^{-1} A \psi - (1 \otimes S_2) \psi \|_{\mathcal{D}(H_0)} = 0$$

for each $\psi \in \mathcal{D}(A; \mathcal{D}(H_0))$ and some operator $S_2 \in \mathcal{B}(\mathcal{H}^2(\mathbb{R}))$. Therefore

$$\| \Lambda^{-2} A^2 \psi - (1 \otimes S_1 S_2) \psi \|_{\mathcal{D}(H_0)} = 0$$

for each $\psi \in \mathcal{D}(A^2; \mathcal{D}(H_0))$. This implies that $\Lambda^{-2} A^2: \mathcal{D}(A^2; \mathcal{D}(H_0)) \rightarrow \mathcal{D}(H_0)$ extends to an element of $\mathcal{B}(\mathcal{D}(H_0))$.

(iii) The short-range decay of V required in [ABG96, Eq. (7.5.29)] follows from Assumption 3.1.

(b) We have $[H_0, A] \in \mathcal{B}(\mathcal{D}(H_0))$ because $[H_0, iA] = 1 \otimes \vartheta(P^2)$ [ABG96, Lemma 7.6.4], [BG92, Sec. 3]. Since $H = H_0 + V$, it remains to show that $[V, A] \in \mathcal{B}(\mathcal{D}(H_0), \mathcal{H})$. This follows by using the fact that V is bounded from $\mathcal{L}^2(\Sigma) \otimes \mathcal{H}_t^{2s}(\mathbb{R})$ to $\mathcal{L}^2(\Sigma) \otimes \mathcal{H}_{t+\kappa}^{2(s-1)}(\mathbb{R})$ for any $s \in [0, 1]$, $t \in \mathbb{R}$, and the fact that A is bounded from $\mathcal{L}^2(\Sigma) \otimes \mathcal{H}_t^s(\mathbb{R})$ to $\mathcal{L}^2(\Sigma) \otimes \mathcal{H}_{t-1}^s(\mathbb{R})$ for any $s, t \in \mathbb{R}$. \square

Since $\{e^{i\tau A}\}_{\tau \in \mathbb{R}}$ leaves $\mathcal{D}(H_0)$ invariant and H_0 is of class $C^\infty(A)$, Lemma 3.2.(a) implies that H is of class $\mathcal{C}^{1,1}(A)$ [ABG96, Thm. 6.3.4.(b)]. This has the following consequence.

Lemma 3.3 *Let V satisfy Assumption 3.1. Then A is conjugate to H on $(-\infty, \nu_1) \cup J_\vartheta$.*

Proof. Since H_0 and H are of class $\mathcal{C}^{1,1}(A)$, $(H+i)^{-1} - (H_0+i)^{-1}$ is compact and A is strictly conjugate to H_0 on $(-\infty, \nu_1) \cup J_\vartheta$, the claim follows by [ABG96, Thm. 7.2.9]. \square

Now we can prove limiting absorption principles for H_0 and H , and state spectral properties of H . If $\mathcal{G}^\mu := \mathcal{D}(H_0^\mu)$, $\mu \in \mathbb{R}$, then the limiting absorption principles can be expressed in terms of the Banach space $\mathcal{K} := (\mathcal{G}^{-1/2} \cap \mathcal{D}(A; \mathcal{G}^{-1}), \mathcal{G}^{-1/2})_{1/2,1}$ defined by real interpolation [ABG96, Chap. 2]. We emphasize that \mathcal{K} contains $\mathcal{L}^2(\Sigma) \otimes \mathcal{H}_t^{-1}(\mathbb{R})$ for any $t > 1/2$, which is shown in the appendix.

Theorem 3.4 *Let V satisfy Assumption 3.1. Then*

- (a) *H has no singularly continuous spectrum.*
- (b) *The eigenvalues of H in $\sigma(H) \setminus \mathcal{T}$ are of finite multiplicity and can accumulate at points of \mathcal{T} only.*
- (c) *The limit $\lim_{\varepsilon \searrow 0} (H_0 - \lambda \mp i\varepsilon)^{-1}$, resp. $\lim_{\varepsilon \searrow 0} (H - \lambda \mp i\varepsilon)^{-1}$, exists in the weak* topology of $\mathcal{B}(\mathcal{K}, \mathcal{K}^*)$ uniformly in λ on each compact subset of $\mathbb{R} \setminus \mathcal{T}$, resp. $\mathbb{R} \setminus (\sigma_p(H) \cup \mathcal{T})$.*

Proof. The operator H is of class $\mathcal{C}^{1,1}(A)$ and A is conjugate to H on $(-\infty, \nu_1) \cup J_\vartheta$ by Lemma 3.3. Furthermore, given any compact set K in $\mathbb{R} \setminus \mathcal{T}$, there exist $\varepsilon \in (0, 1)$ and $\vartheta \in C_0^\infty((\varepsilon, \infty))$ such that K is contained in $(-\infty, \nu_1) \cup J_\vartheta$. Therefore the assertions (a) and (b) follow by the conjugate operator method [ABG96, Cor. 7.2.11 & Thm. 7.4.2]. Due to Lemma 3.2.(b) and the regularity properties of H_0 and H with respect to A , the limiting absorption principles are obtained via [ABG96, Thm. 7.5.2]. \square

Corollary 3.5 *Let V satisfy Assumption 3.1. Then*

- (a) *If T belongs to $\mathcal{B}(\mathcal{L}^2(\Sigma) \otimes \mathcal{H}_{-t}^1(\mathbb{R}), \mathcal{H})$ for some $t > 1/2$, then T is locally H_0 -smooth (resp. H -smooth) on $\mathbb{R} \setminus \mathcal{T}$ (resp. $\mathbb{R} \setminus (\sigma_p(H) \cup \mathcal{T})$).*
- (b) *The wave operators W^\pm exist and are complete.*

Proof. (a) Let $\mathcal{E} := \mathcal{L}^2(\Sigma) \otimes \mathcal{H}_t^{-1}(\mathbb{R})$. Since $\mathcal{E} \subset \mathcal{D}(H_0)^*$ densely, and $\mathcal{E} \subset \mathcal{K}$, it is enough to verify the remaining hypothesis of [ABG96, Prop. 7.1.3.(b)] on \mathcal{E} to prove the statement. Let $\mathcal{E}^{*\circ}$ be the closure of $\mathcal{D}(H_0)$ in \mathcal{E}^* , equipped with the norm of \mathcal{E}^* . Clearly $\mathcal{E}^{*\circ} \subset \mathcal{E}^*$. Furthermore, since $\mathcal{D}(H_0)$ is dense in \mathcal{E}^* , we

also have $\mathcal{E}^* \subset \mathcal{E}^{*\circ}$. Therefore $\mathcal{E}^* = \mathcal{E}^{*\circ}$. By taking the adjoint, this leads to $\mathcal{E} = (\mathcal{E}^{*\circ})^*$.

(b) By the point (a), $V_1 := 1 \otimes \langle Q \rangle^{-\kappa/2} \langle P \rangle$ is locally H_0 -smooth on $\mathbb{R} \setminus \mathcal{T}$ and $V_2 := (1 \otimes \langle Q \rangle^{\kappa/2} \langle P \rangle^{-1})V$ is locally H -smooth on $\mathbb{R} \setminus (\sigma_p(H) \cup \mathcal{T})$. Since $\sigma_p(H) \cup \mathcal{T}$ is countable and $\langle \varphi, V\psi \rangle = \langle V_1\varphi, V_2\psi \rangle$ for all $\varphi, \psi \in \mathcal{D}(H_0)$, one can conclude by applying the smooth perturbation theory [RS78, Corollary to Thm. XIII.31]. \square

Under Assumption 3.1 one could also find optimal spaces where the analogue of the limiting absorption principles of Theorem 3.4.(c) holds in norm. The following particular result is sufficient for us. If $t > 1/2$, then the boundary values

$$R^{H_0}(\lambda \pm i0) := \lim_{\varepsilon \searrow 0} (H_0 - \lambda \mp i\varepsilon)^{-1}, \quad \lambda \in \mathbb{R} \setminus \mathcal{T},$$

and

$$R^H(\lambda \pm i0) := \lim_{\varepsilon \searrow 0} (H - \lambda \mp i\varepsilon)^{-1}, \quad \lambda \in \mathbb{R} \setminus (\sigma_p(H) \cup \mathcal{T}),$$

exist in $\mathcal{B}(\mathbf{L}^2(\Sigma) \otimes \mathcal{H}_t(\mathbb{R}), \mathbf{L}^2(\Sigma) \otimes \mathcal{H}_{-t}(\mathbb{R}))$ (see [BGM93, Thm. 4.13]). In the rest of the section we study the norm differentiability of the function $\lambda \mapsto S(\lambda)$, which relies on the differentiability of the function $\lambda \mapsto R^H(\lambda \pm i0)$.

Lemma 3.6 *Let $t > n + 1/2$, $n \in \mathbb{N}$. Let V satisfy Assumption 3.1 with $\kappa > n + 1$. Then $\lambda \mapsto R^H(\lambda + i0)$ is n times continuously differentiable as a map from $(\nu_1, \infty) \setminus (\sigma_p(H) \cup \mathcal{T})$ to $\mathcal{B}(\mathbf{L}^2(\Sigma) \otimes \mathcal{H}_t(\mathbb{R}), \mathbf{L}^2(\Sigma) \otimes \mathcal{H}_{-t}(\mathbb{R}))$.*

Proof. Since H_0 is of class $C^\infty(A)$ and $\mathbf{L}^2(\Sigma) \otimes \mathcal{H}_t(\mathbb{R}) \subset \mathcal{D}(\langle A \rangle^t)$, we have the following result [BGS, Sec. 1.7]. For each $\lambda \in (\nu_1, \infty) \setminus \mathcal{T}$ and $k \leq n$, the boundary values $\lim_{\varepsilon \searrow 0} (H_0 - \lambda \mp i\varepsilon)^{-k-1}$ exist in $\mathcal{B}(\mathbf{L}^2(\Sigma) \otimes \mathcal{H}_t(\mathbb{R}), \mathbf{L}^2(\Sigma) \otimes \mathcal{H}_{-t}(\mathbb{R}))$. Furthermore $\lambda \mapsto R^{H_0}(\lambda \pm i0)$ is k times continuously differentiable as a map from $(\nu_1, \infty) \setminus \mathcal{T}$ to $\mathcal{B}(\mathbf{L}^2(\Sigma) \otimes \mathcal{H}_t(\mathbb{R}), \mathbf{L}^2(\Sigma) \otimes \mathcal{H}_{-t}(\mathbb{R}))$ with

$$\frac{d^k}{d\lambda^k} R^{H_0}(\lambda \pm i0) = k! \lim_{\varepsilon \searrow 0} (H_0 - \lambda \mp i\varepsilon)^{-k-1}.$$

Thus one can apply the inductive method of [JN92, Lemma 4.3] to infer the result for H from the one for H_0 . \square

In the following lemma we prove the usual formula for the S -matrix.

Lemma 3.7 *Let V satisfy Assumption 3.1. Then for each $\lambda \in (\nu_1, \infty) \setminus (\sigma_p(H) \cup \mathcal{T})$, one has the equality*

$$S(\lambda) = 1 - i\pi T(\lambda) (1 \otimes \mathcal{F}) [1 - V R^H(\lambda + i0)] V (1 \otimes \mathcal{F}^*) T(\lambda)^*. \quad (3.2)$$

Proof. The claim is a consequence of the stationary method [Kur73, Thm. 6.3] applied to the pair $\{H_0, H\}$. Therefore we simply verify the principal hypotheses of that theorem.

The total Hamiltonian admits the factorization $H = H_0 + V_1 V_2$ where V_1 is the H_0 -compact operator $1 \otimes \langle Q \rangle^{-\kappa/2}$ (see [KT04, Lemma 2.1]) and V_2 is the (maximal) operator associated to $1 \otimes \langle Q \rangle^{\kappa/2} V$. Moreover, since $T: (\nu_1, \infty) \setminus \mathcal{T} \rightarrow \mathcal{B}(\mathcal{L}^2(\Sigma) \otimes \mathcal{H}_t^s(\mathbb{R}), \mathcal{H}(\infty))$ is locally Hölder continuous for each $t \in \mathbb{R}$, $s > 1/2$, the functions $T(\cdot; V_j): (\nu_1, \infty) \setminus \mathcal{T} \rightarrow \mathcal{B}(\mathcal{H}, \mathcal{H}(\infty))$, $j = 1, 2$, defined by

$$T(\lambda; V_j)\varphi := (\mathcal{U}V_j^*\varphi)(\lambda),$$

are locally Hölder continuous. \square

Finally we have the following result on the norm differentiability of the function $\lambda \mapsto S(\lambda)$.

Proposition 3.8 *Let V satisfy Assumption 3.1 with $\kappa > n + 1$, $n \in \mathbb{N}$. Then $\lambda \mapsto S(\lambda)$ is n times continuously differentiable as a map from $(\nu_1, \infty) \setminus (\sigma_p(H) \cup \mathcal{T})$ to $\mathcal{H}(\infty)$.*

Proof. Due to (3.2) and Lemmas 2.4.(c) and 3.6, all operators in the expression for $S(\lambda)$ are n times continuously norm differentiable. Then a direct calculation as in the proof of [Jen81, Thm. 3.5] implies the claim. \square

3.2 Existence theorem

To illustrate Theorem 1.2, we verify in this section the existence of the (global) time delay in the case $H := H_0 + V$, where V satisfies Assumption 3.1 with $\kappa > 4$. To begin with we prove two technical lemmas in relation with the hypotheses of Theorem 1.2.

Lemma 3.9 *If V satisfies Assumption 3.1 with $\kappa > 2$ and $\varphi \in \mathcal{D}_\tau^\Omega$ for some $\tau > 2$, then*

$$\|(W^- - 1)e^{-itH_0}\varphi\| \in \mathcal{L}^1((-\infty, 0), dt) \quad (3.3)$$

and

$$\|(W^+ - 1)e^{-itH_0}\varphi\| \in \mathcal{L}^1((0, \infty), dt). \quad (3.4)$$

Proof. For $\varphi \in \mathcal{D}_\tau^\Omega$ and $t \in \mathbb{R}$, we have (see the proof of [Jen81, Lemma 4.6])

$$(W^- - 1)e^{-itH_0}\varphi = -ie^{-itH} \int_{-\infty}^t ds e^{isH} V e^{-isH_0}\varphi,$$

where the integral is strongly convergent. Hence to prove (3.3) it is enough to show that

$$\int_{-\infty}^{-\delta} dt \int_{-\infty}^t ds \|Ve^{-isH_0}\varphi\| < \infty \quad (3.5)$$

for some $\delta > 0$. We know from Remark 2.1 that $\varphi = \sum_{\alpha \leq \text{Const.}} \varphi_\alpha^\Sigma \otimes \varphi_\alpha^\mathbb{R}$, where $\varphi_\alpha^\Sigma \in \mathcal{P}_\alpha \mathcal{L}^2(\Sigma)$ and $\varphi_\alpha^\mathbb{R} \in \mathcal{D}_\tau^\mathbb{R}$. Thus there exists $\eta \in C_0^\infty((0, \infty))$ such that $1 \otimes$

$\eta(P^2)\varphi = \varphi$. Furthermore, if $\zeta := \min\{\kappa, \tau\}$, then $\|\langle Q \rangle^\zeta \varphi_\alpha^\mathbb{R}\| < \infty$ and $V(1 \otimes \langle P \rangle^{-2} \langle Q \rangle^\zeta)$ belongs to $\mathcal{B}(\mathcal{H})$ due to Assumption 3.1. This implies that

$$\begin{aligned} & \|V e^{-isH_0} \varphi\| \\ & \leq \sum_{\alpha \leq \text{Const.}} \|V(1 \otimes \langle P \rangle^{-2} \langle Q \rangle^\zeta) [\varphi_\alpha^\Sigma \otimes \langle Q \rangle^{-\zeta} \langle P \rangle^2 \eta(P^2) e^{-isP^2} \langle Q \rangle^{-\zeta} \langle Q \rangle^\zeta \varphi_\alpha^\mathbb{R}]\| \\ & \leq \text{Const.} \|\langle Q \rangle^{-\zeta} \langle P \rangle^2 \eta(P^2) e^{-isP^2} \langle Q \rangle^{-\zeta}\|. \end{aligned}$$

For each $\varepsilon > 0$, it follows from [ACS87, Lemma 9] that there exists a constant $C > 0$ such that $\|V e^{-isH_0} \varphi\| \leq C(1 + |s|)^{-\zeta + \varepsilon}$. Since $\zeta > 2$, this implies (3.5). The proof of (3.4) is similar. \square

Let \mathcal{E} be the finite span of vectors $\varphi \in \mathcal{H}$ of the form $\{\varphi(\lambda)\} = \{\rho(\lambda)h(\lambda)\}$ in the spectral representation of H_0 , where $\rho: (\nu_1, \infty) \rightarrow \mathbb{C}$ is three times continuously differentiable and has compact support in $(\nu_1, \infty) \setminus (\sigma_p(H) \cup \mathcal{T})$, and $\lambda \mapsto h(\lambda) \in \mathcal{H}(\lambda)$ is λ -independent on each interval $(\nu_\alpha, \nu_{\alpha+1})$. Clearly the set \mathcal{E} is dense in \mathcal{H} . Furthermore one has the following inclusions.

Lemma 3.10

- (a) \mathcal{E} is contained in \mathcal{D}_3^Ω .
- (b) Let V satisfy Assumption 3.1 with $\kappa > 4$. Then $S\mathcal{E}$ is contained in \mathcal{D}_3^Ω .

Proof. (a) Let $\varphi \in \mathcal{E}$. It is clear that there exists a compact set J in $(\nu_1, \infty) \setminus (\sigma_p(H) \cup \mathcal{T})$ such that $E^{H_0}(J)\varphi = \varphi$. Thus, in order to show that $\varphi \in \mathcal{D}_3^\Omega$, one has to verify that $\varphi \in \mathbf{L}^2(\Sigma) \otimes \mathcal{H}_3(\mathbb{R}) = \mathcal{D}(1 \otimes Q^3)$.

Let $\psi \in \mathbf{L}^2(\Sigma) \odot \mathcal{S}(\mathbb{R})$. Then, using (2.3)–(2.5), we obtain

$$[\mathcal{U}(1 \otimes Q^3)\psi]_\alpha(\lambda) = \{ig_\alpha^-(\lambda), -ig_\alpha^+(\lambda)\}, \quad (3.6)$$

where

$$\begin{aligned} g_\alpha^\pm(\lambda) &:= \frac{3}{8}(\lambda - \nu_\alpha)^{-3/2}(\mathcal{U}\psi)_\alpha^\pm(\lambda) + \frac{3}{2}(\lambda - \nu_\alpha)^{-1/2} \frac{d}{d\lambda}(\mathcal{U}\psi)_\alpha^\pm(\lambda) \\ &+ 18(\lambda - \nu_\alpha)^{1/2} \frac{d^2}{d\lambda^2}(\mathcal{U}\psi)_\alpha^\pm(\lambda) + 8(\lambda - \nu_\alpha)^{3/2} \frac{d^3}{d\lambda^3}(\mathcal{U}\psi)_\alpha^\pm(\lambda). \end{aligned} \quad (3.7)$$

The r.h.s. of (3.6)–(3.7) with $\psi \in \mathbf{L}^2(\Sigma) \odot \mathcal{S}(\mathbb{R})$ replaced by $\varphi \in \mathcal{E}$ defines a vector $\tilde{\varphi}$ belonging to $\int_{[\nu_1, \infty)}^\oplus d\lambda \mathcal{H}(\lambda)$. Thus, using partial integration for the terms involving derivatives with respect to λ , one finds that

$$|\langle (1 \otimes Q^3)\psi, \varphi \rangle| = |\langle \mathcal{U}\psi, \tilde{\varphi} \rangle| \leq \text{Const.} \|\psi\|$$

for all $\psi \in \mathbf{L}^2(\Sigma) \odot \mathcal{S}(\mathbb{R})$, $\varphi \in \mathcal{E}$. Since $(1 \otimes Q^3) \upharpoonright \mathbf{L}^2(\Sigma) \odot \mathcal{S}(\mathbb{R})$ is essentially selfadjoint, this implies that $\varphi \in \mathcal{D}(1 \otimes Q^3)$.

(b) By Proposition 3.8 the function $\lambda \mapsto S(\lambda)$ is three times continuously norm differentiable. Thus the argument in point (a) with φ replaced by $S\varphi$ gives the result. \square

Theorem 3.11 *Let $H := H_0 + V$, where V satisfies Assumption 3.1 with $\kappa > 4$. Then, for each $\varphi \in \mathcal{E}$, $\tau_r(\varphi)$ exists for all $r > 0$ and $\tau_r(\varphi)$ converges as $r \rightarrow \infty$ to a finite limit equal to $\langle \varphi, \tau_{E-W} \varphi \rangle$.*

Proof. We apply Theorem 1.2. The hypotheses 1 and 2 of that theorem are satisfied due to Corollary 3.5, and the hypotheses on $\varphi \in \mathcal{E}$ follow from Lemmas 3.9 and 3.10. Since the function $\lambda \mapsto S(\lambda)$ is strongly continuously differentiable on $(\nu_1, \infty) \setminus (\sigma_p(H) \cup \mathcal{T})$, the proof is complete. \square

Appendix

Proof of Lemma 2.4. (a) Fix $\lambda \in (\nu_1, \infty) \setminus \mathcal{T}$ and let $\varphi \in \mathbb{L}^2(\Sigma) \odot \mathcal{S}(\mathbb{R})$. Choose $f \in C_0^\infty(\mathbb{R})$ such that

$$[1 \otimes \gamma(\pm \sqrt{\lambda - \nu_\alpha})] \varphi = [1 \otimes \gamma(\pm \sqrt{\lambda - \nu_\alpha})] [1 \otimes f(Q)] \varphi$$

for each $\alpha \in \mathbb{N}(\lambda)$. Then we get

$$\begin{aligned} \|T(\lambda)\varphi\|_{\mathcal{H}(\infty)}^2 &\leq \text{Const.} \sum_{\alpha \in \mathbb{N}(\lambda)} \left\{ \left\| [1 \otimes \gamma(-\sqrt{\lambda - \nu_\alpha})] f(Q) \right\|_{\mathbb{L}^2(\Sigma)}^2 \right. \\ &\quad \left. + \left\| [1 \otimes \gamma(\sqrt{\lambda - \nu_\alpha})] f(Q) \right\|_{\mathbb{L}^2(\Sigma)}^2 \right\}. \end{aligned}$$

Since $\gamma(\pm \sqrt{\lambda - \nu_\alpha})$ extends to an element of $\mathcal{B}(\mathcal{H}^s(\mathbb{R}), \mathbb{C})$ [Kur78, Thm. 2.4.2] and $f(Q)$ is bounded from $\mathcal{H}_t^s(\mathbb{R})$ to $\mathcal{H}^s(\mathbb{R})$, this implies that

$$\|T(\lambda)\varphi\|_{\mathcal{H}(\infty)}^2 = \text{Const.} \|\varphi\|_{\mathbb{L}^2(\Sigma) \otimes \mathcal{H}_t^s(\mathbb{R})}^2.$$

(b) Let K be a compact set in $(\nu_1, \infty) \setminus \mathcal{T}$. Choose $\delta = \delta(K) > 0$ such that λ_1 and λ_2 belong to the same interval $(\nu_\alpha, \nu_{\alpha+1})$ whenever $\lambda_1, \lambda_2 \in K$ and $|\lambda_1 - \lambda_2| < \delta$. Let $\varphi \in \mathbb{L}^2(\Sigma) \odot \mathcal{S}(\mathbb{R})$. Due to the point (a), it is enough to show that there exist $\zeta > 0$ such that

$$\|[T(\lambda_1) - T(\lambda_2)]\varphi\|_{\mathcal{H}(\infty)} \leq \text{Const.} |\lambda_1 - \lambda_2|^\zeta \|\varphi\|_{\mathbb{L}^2(\Sigma) \otimes \mathcal{H}_t^s(\mathbb{R})} \quad (3.8)$$

if $\lambda_1, \lambda_2 \in K$ and $|\lambda_1 - \lambda_2| < \delta$.

Choose $f \in C_0^\infty(\mathbb{R} \setminus \{0\})$ such that

$$(\lambda - \nu_\alpha)^{-\frac{1}{4}} [1 \otimes \gamma(\pm \sqrt{\lambda - \nu_\alpha})] \varphi = [1 \otimes \gamma(\pm \sqrt{\lambda - \nu_\alpha})] [1 \otimes |Q|^{-\frac{1}{2}} f(Q)] \varphi$$

for each $\lambda \in K$, $\alpha \in \mathbb{N}(\sup K)$. Then we get

$$\begin{aligned} &\|[T(\lambda_1) - T(\lambda_2)]\varphi\|_{\mathcal{H}(\infty)}^2 \\ &\leq \text{Const.} \sum_{\alpha \in \mathbb{N}(\lambda_1)} \left\{ \left\| 1 \otimes [(\gamma(-\sqrt{\lambda_1 - \nu_\alpha}) - \gamma(-\sqrt{\lambda_2 - \nu_\alpha})) |Q|^{-\frac{1}{2}} f(Q)] \right\|_{\mathbb{L}^2(\Sigma)}^2 \right. \\ &\quad \left. + \left\| 1 \otimes [(\gamma(\sqrt{\lambda_1 - \nu_\alpha}) - \gamma(\sqrt{\lambda_2 - \nu_\alpha})) |Q|^{-\frac{1}{2}} f(Q)] \right\|_{\mathbb{L}^2(\Sigma)}^2 \right\}. \end{aligned}$$

Since the function $\mathbb{R} \ni \xi \mapsto \gamma(\xi) \in \mathcal{B}(\mathcal{H}^s(\mathbb{R}), \mathbb{C})$ is Hölder continuous [Kur78, Thm. 2.4.2] and $|Q|^{-\frac{1}{2}}f(Q)$ is bounded from $\mathcal{H}_t^s(\mathbb{R})$ to $\mathcal{H}^s(\mathbb{R})$, this implies (3.8).

(c) The proof is similar to that of [Jen81, Lemma 3.3]. \square

Proof of the embedding $\mathcal{L}^2(\Sigma) \otimes \mathcal{H}_t^{-1}(\mathbb{R}) \subset \mathcal{K}$ for any $t > 1/2$. Since $\mathcal{D}(A; \mathcal{G}^{-1/2}) \subset \mathcal{G}^{-1/2} \cap \mathcal{D}(A; \mathcal{G}^{-1})$ and $\mathcal{L}^2(\Sigma) \otimes \mathcal{H}^{-1}(\mathbb{R}) \subset \mathcal{G}^{-1/2}$, we have $(\mathcal{D}[A; \mathcal{L}^2(\Sigma) \otimes \mathcal{H}^{-1}(\mathbb{R})], \mathcal{L}^2(\Sigma) \otimes \mathcal{H}^{-1}(\mathbb{R}))_{1/2,1} \subset \mathcal{K}$ due to [ABG96, Cor. 2.6.3]. Then we obtain that $(\mathcal{D}[A; \mathcal{L}^2(\Sigma) \otimes \mathcal{H}^{-1}(\mathbb{R})], \mathcal{L}^2(\Sigma) \otimes \mathcal{H}^{-1}(\mathbb{R}))_{\mu,2} \subset \mathcal{K}$ for any $\mu < 1/2$, by using [ABG96, Thm. 3.4.3.(a)]. Since $\mathcal{L}^2(\Sigma) \otimes \mathcal{H}_1^{-1}(\mathbb{R}) \subset \mathcal{D}[A; \mathcal{L}^2(\Sigma) \otimes \mathcal{H}^{-1}(\mathbb{R})]$, this leads to the embedding $(\mathcal{L}^2(\Sigma) \otimes \mathcal{H}_1^{-1}(\mathbb{R}), \mathcal{L}^2(\Sigma) \otimes \mathcal{H}^{-1}(\mathbb{R}))_{\mu,2} \subset \mathcal{K}$ [ABG96, Cor. 2.6.3]. Now, by using [Aub00, Thm. 12.6.1] and [LP64, Thm. VII(I.1)], we get the isometry $\mathcal{L}^2(\Sigma) \otimes \mathcal{H}_{1-\mu}^{-1}(\mathbb{R}) \simeq (\mathcal{L}^2(\Sigma) \otimes \mathcal{H}_1^{-1}(\mathbb{R}), \mathcal{L}^2(\Sigma) \otimes \mathcal{H}^{-1}(\mathbb{R}))_{\mu,2}$. Therefore $\mathcal{L}^2(\Sigma) \otimes \mathcal{H}_t^{-1}(\mathbb{R}) \subset \mathcal{K}$ for any $t > 1/2$. \square

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